Simple Harmonic Motion

A pendulum swinging back and forth or a mass oscillating on a spring are two examples of Simple Harmonic Motion (SHM.) SHM occurs any time the position of an object as a function of time can be represented by a sine wave. We would then write the position as a function of time as

$$x = A\sin(\omega t + \varphi)$$

where t is time, A is the amplitude of the oscillations, φ is called the "phase shift" of the motion, and is simply a constant that indicates the initial position of the object. Angles are measured in radians. The variable ω is called the angular frequency. This can be seen by realizing that the sine function repeats every 2π. Calling the period of oscillation T, we can write

$$(\omega(t+T)+\varphi) = (\omega t + \varphi) + 2\pi$$

$$\omega T = 2\pi$$

which means that

$$T = \frac{2\pi}{\omega}$$

The velocity and acceleration of the object are found by taking the first and second derivative of the position:

$$v = \omega A\cos(\omega t + \varphi)$$

$$a = -\omega^2 A\sin(\omega t + \varphi)$$

These are often written in "dot" form as

$$\dot{x} = \omega A\cos(\omega t + \varphi)$$

$$\ddot{x} = -\omega^2 A\sin(\omega t + \varphi)$$

Notice that the acceleration function is simply the position function multiplied by a constant, so that we can write

$$\ddot{x} = -\omega^2 x$$

This can be generalized so that anytime the acceleration is proportional to the position, the object is undergoing simple harmonic motion with period T and angular frequency ω, as given above.

Notice in the expressions above that we can also say the following:

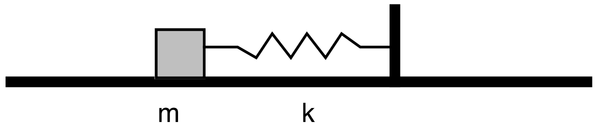
$$Displacement_{\max} = A$$

$$Speed_{\max} = A\omega$$

$$Acceleration_{\max} = A\omega^2$$

Example 1: Mass on a spring

Consider a mass m on a horizontal frictionless table attached to a spring with a spring constant k.



The spring can both stretch and contract, and the position x is defined to be zero at the "unstretched" position shown. Since we are assuming a frictionless table, the only force on the spring would be because of any stretching that occurs in the spring (the normal force and weight would cancel.) So the net force on the mass is simply Hooke's Law: F = –kx. The negative sign is actually very important. It shows that the force on the mass is always opposite its displacement, meaning when the mass is on the right (+x) the force is to the left (–) and vice versa. Worded another way, the net force on the mass is always trying to get the mass to the equlibrium position, which is x=0.

So continuing on with Newton's Second Law and Hooke's Law and doing a tiny bit of algebra we have

$$F = -kx$$
$$ma = -kx$$
$$m\ddot{x} = -kx$$
$$\ddot{x} = -\frac{k}{m}x$$

This is the equation for simple harmonic motion derived earlier! Therefor, the angular frequency of the object on the spring is

$$\omega^2 = \frac{k}{m}$$
$$\omega = \sqrt{\frac{k}{m}}$$

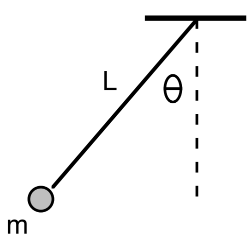
The period of the object on the spring is related to the angular frequency so

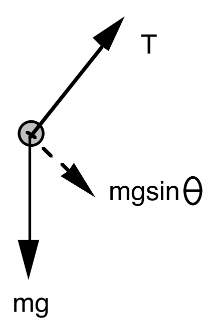
$$T = \frac{2\pi}{\omega}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$

This equation is also valid for a mass hanging from a spring.

Example 2: A Simple Pendulum

Consider a mass m hanging from pendulum of length L. To find an expression for the period of this pendulum, we will go through an analysis like we did for the spring: apply Newton's Laws, look for the simple harmonic relationship to find the angular frequency and solve for the period. Below is a diagram of the pendulum and the free-body diagram for the mass.





There are only two forces acting on the mass as it falls, the tension in the string and the weight of the mass. The net force on the mass varies with time, and always has two components to it. The net force component that is parallel to the string is the instantaneous centripetal acceleration of the mass, and the component that is perpendicular to the string is the instantaneous tangential acceleration. Note that the centripetal force is greatest at the bottom of the swing when the tangential acceleration is zero (no horizontal component of weight), and the tangential acceleration is greatest at the top of its swing, when the centripetal force is zero (v=0).

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The position of the mass would be given by the angle θ alone, since we are assuming the string does not stretch in this process, and so we can ignore the centripetal component of the net force. The linear force on the mass is thus

$$F = -mg\sin\theta$$

$$ma = -mg\sin\theta$$

$$a = -g\sin\theta$$

The minus sign is because the direction of the acceleration is opposite the direction of the position. The tangential acceleration is related to the angular acceleration so that

$$a = L\alpha = -g\sin\theta$$

which can be rewritten as

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

Now we make an approximation. For small angles, $\sin\theta \approx \theta$ (remember that this is in radians. This is a common approximation in physics. See Note 2 at end.)

So we get

$$\ddot{\theta} = -\frac{g}{L}\theta$$

which is the simple harmonic motion equation with

$$\omega^2 = \frac{g}{L}$$

The period of the pendulum is therefore

$$T = \frac{2\pi}{\omega}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$

This holds true for small angles. For small angular displacements, the period of a pendulum depends only on the length of the pendulum and the acceleration due to gravity.

Example 3: The Physical Pendulum

Any body that is hanging and swinging is a pendulum, but the equation derived above is only valid for simple pendulums, a mass swinging on a "massless" string. A physical pendulum simply refers to any rigid body that is swinging back and forth, and we will derive an expression for its period in the same manner as before, except that now we will be applying Newton's Laws in rotational form. We will again be making the small angle approximation, but the only other restriction is that the body is a rigid one (and we will see that the simple pendulum equation already derived is just a special case of the physical pendulum.

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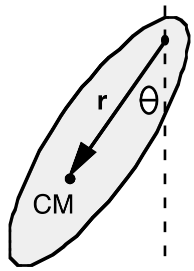
We start off with a rigid body hanging from a point that is not its center of mass (CM). This will produce a torque on the body, causing it to swing back and forth. Note that if the body was hanging from the center of mass, there would be no net torque on the body, and it would not swing.



The net torque on the body will be because of gravity acting on the center of mass so that

$$\sum \tau = r \times F = I\alpha$$

$$rmg \sin\theta = I\alpha$$

Rewriting the equation in dot form, and making the small angle approximation gives us

$$\ddot{\theta} = -\frac{rmg}{I}\theta$$

Which is the simple harmonic motion equation. Solving for the period, we then get

$$T = 2\pi\sqrt{\frac{I}{rmg}}$$

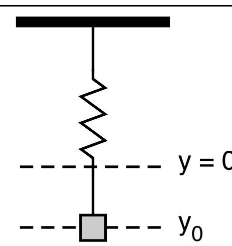
where I is the moment of inertia, m the mass, g the acceleration due to gravity and r is the distance from the point of oscillation to the center of mass. In the case of the simple pendulum, r is the length of the pendulum, L, and I is mL^2. Substituting this into the equation just derived yields

$$T = 2\pi\sqrt{\frac{mL^2}{Lmg}} = 2\pi\sqrt{\frac{L}{g}}$$

which is what we found in Example 2.

Note 1: Vertical Spring

Hanging a mass vertically has the exact same period of oscillation. To see this, consider a mass hanging from a spring. We will define y=0 to be the position of the mass when the spring is unstretched, y_0 to be the equilibrium position of the mass, and down to be positive. By definition, this equilibrium position is when the weight of the mass is balanced by the spring force, or



$$mg = ky_0$$

Now if we displace the spring a little, there is a net force on the mass trying to return it to its equilibrium position, so that

$$\sum F = ma = mg - ky$$

$$m\ddot{y} = mg - ky$$

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This almost looks like the Simple Harmonic Equation, except for the constant weight term. This can be taken care of by changing the coordinates. Let's define

$$x = y - y_0$$

so that the variable x is zero at the equilibrium position, and is positive when the mass is below this point, and negative when above it. From that definition, we can take the first and second derivatives so that

$$\dot{x} = \dot{y}$$

$$\ddot{x} = \ddot{y}$$

Now substitute x's for all the y's in the equation above and doing a little algebra gives us

$$m\ddot{y} = mg - ky$$

$$m\ddot{x} = mg - k(x + y_0)$$

$$m\ddot{x} = -kx + mg - ky_0$$

Since $mg = ky_0$, this reduces to

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

which is the Simple Harmonic Equation we derived earlier. It doesn't matter if a mass on a spring is hanging vertically, at an angle or is horizontal; its period of oscillation will be given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Note 2: Small Angle Approximation

The small angle approximation tends to annoy people the first time they see it. To help see how valid it is, the chart below shows values of θ and sinθ. for a few angles. The approximation works pretty well (< 5% error) for angles smaller than 30 degrees, and very well (< 1% error) for angles smaller than 10 degrees.

θ (radians)	sin θ	% difference	θ (degrees)
0.000	0.000	0.0	0.0
0.100	0.100	0.2	5.7
0.200	0.199	0.7	11.5
0.300	0.296	1.5	17.2
0.400	0.389	2.7	22.9
0.500	0.479	4.3	28.7
0.600	0.565	6.3	34.4
0.700	0.644	8.7	40.1
0.800	0.717	11.5	45.9
0.900	0.783	14.9	51.6
1.000	0.841	18.8	57.3
1.100	0.891	23.4	63.1
1.200	0.932	28.7	68.8
1.300	0.964	34.9	74.5
1.400	0.985	42.1	80.3
1.500	0.997	50.4	86.0